ESTIMATES FOR THE VOLUME OF A LORENTZIAN MANIFOLD

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Abstract. We prove new estimates for the volume of a Lorentzian manifold and show especially that cosmological spacetimes with crushing singularities have finite volume.

0. Introduction

Let N be a $(n+1)$-dimensional Lorentzian manifold and suppose that N can be decomposed in the form

$$N = N_0 \cup N_- \cup N_+, \tag{0.1}$$

where N_0 has finite volume and N_- resp. N_+ represent the critical past resp. future Cauchy developments with not necessarily a priori bounded volume. We assume that N_+ is the future Cauchy development of a Cauchy hypersurface M_1, and N_- the past Cauchy development of a hypersurface M_2, or, more precisely, we assume the existence of a time function x^0, such that

$$\begin{aligned} N_+ &= {x^0}^{-1}([t_1, T_+)), \qquad & M_1 &= \{x^0 = t_1\}, \\ N_- &= {x^0}^{-1}((T_-, t_2]), \qquad & M_2 &= \{x^0 = t_2\}, \end{aligned} \tag{0.2}$$

and that the Lorentz metric can be expressed as

$$d\bar{s}^2 = e^{2\psi}\{-{dx^0}^2 + \sigma_{ij}(x^0, x)dx^i dx^j\}, \tag{0.3}$$

where $x = (x^i)$ are local coordinates for the space-like hypersurface M_1 if N_+ is considered resp. M_2 in case of N_-.

The coordinate system $(x^\alpha)_{0\le\alpha\le n}$ is supposed to be future directed, i.e. the *past* directed unit normal (ν^α) of the level sets

$$M(t) = \{x^0 = t\} \tag{0.4}$$

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is of the form

$$(\nu^\alpha) = -e^{-\psi}(1, 0, \dots, 0). \tag{0.5}$$

If we assume the mean curvature of the slices $M(t)$ with respect to the past directed normal—cf. [5, Section 2] for a more detailed explanation of our conventions—is strictly bounded away from zero, then, the following volume estimates can be proved

Theorem 0.1. *Suppose there exists a positive constant ϵ_0 such that*

$$H(t) \ge \epsilon_0 \qquad \forall\, t_1 \le t < T_+, \tag{0.6}$$

and

$$H(t) \le -\epsilon_0 \qquad \forall\, T_- < t \le t_2, \tag{0.7}$$

then

$$|N_+| \le \frac{1}{\epsilon_0}|M(t_1)|, \tag{0.8}$$

and

$$|N_-| \le \frac{1}{\epsilon_0}|M(t_2|. \tag{0.9}$$

These estimates also hold locally, i.e. if $E_i \subset M(t_i)$, $i = 1, 2$, are measurable subsets and E_1^+, E_2^- the corresponding future resp. past directed cylinders, then,

$$|E_1^+| \le \frac{1}{\epsilon_0}|E_1|, \tag{0.10}$$

and

$$|E_2^-| \le \frac{1}{\epsilon_0}|E_2|. \tag{0.11}$$

1. Proof of Theorem 0.1

In the following we shall only prove the estimate for N_+, since the other case N_- can easily be considered as a future development by reversing the time direction.

Let $x = x(\xi)$ be an embedding of a space-like hypersurface and (ν^α) be the past directed normal. Then, we have the Gauß formula

$$x^\alpha_{ij} = h_{ij}\nu^\alpha. \tag{1.1}$$

where (h_{ij}) is the second fundamental form, and the Weingarten equation

$$\nu_i^\alpha = h_i^k x_k^\alpha. \tag{1.2}$$

We emphasize that covariant derivatives, indicated simply by indices, are always *full* tensors.

The slices $M(t)$ can be viewed as special embeddings of the form

$$x(t) = (t, x^i), \tag{1.3}$$

where (x^i) are coordinates of the *initial* slice $M(t_1)$. Hence, the slices $M(t)$ can be considered as the solution of the evolution problem

$$\dot{x} = -e^\psi \nu, \qquad t_1 \le t < T_+, \tag{1.4}$$

with initial hypersurface $M(t_1)$, in view of (0.5).

From the equation (1.4) we can immediately derive evolution equations for the geometric quantities g_{ij}, h_{ij}, ν, and $H = g^{ij} h_{ij}$ of $M(t)$, cf. e.g. [3, Section 4], where the corresponding evolution equations are derived in Riemannian space.

For our purpose, we are only interested in the evolution equation for the metric, and we deduce

$$\dot{g}_{ij} = \langle \dot{x}_i, x_j \rangle + \langle x_i, \dot{x}_j \rangle = -2e^\psi h_{ij}, \tag{1.5}$$

in view of the Weingarten equation.

Let $g = \det(g_{ij})$, then,

$$\dot{g} = g g^{ij} \dot{g}_{ij} = -2e^\psi H g, \tag{1.6}$$

and thus, the volume of $M(t)$, $|M(t)|$, evolves according to

$$\frac{d}{dt}|M(t)| = \int_{M(t_1)} \frac{d}{dt}\sqrt{g} = -\int_{M(t)} e^\psi H, \tag{1.7}$$

where we shall assume without loss of generality that $|M(t_1|$ is finite, otherwise, we replace $M(t_1)$ by an arbitrary measurable subset of $M(t_1)$ with finite volume.

Now, let $T \in [t_1, T_+)$ be arbitrary and denote by $Q(t_1, T)$ the cylinder

$$Q(t_1, T) = \{\, (x^0, x) \colon t_1 \le x^0 \le T \,\}, \tag{1.8}$$

then,

$$|Q(t_1,T)| = \int_{t_1}^{T}\int_M e^{\psi}, \tag{1.9}$$

where we omit the volume elements, and where, $M = M(x^0)$.

By assumption, the mean curvature H of the slices is bounded from below by ϵ_0, and we conclude further, with the help of (1.7),

$$\begin{aligned} |Q(t_1,T)| &\le \frac{1}{\epsilon_0}\int_{t_1}^{T}\int_M e^{\psi} H \\ &= \frac{1}{\epsilon_0}\{|M(t_1)| - |M(T)|\} \\ &\le \frac{1}{\epsilon_0}|M(t_1)|. \end{aligned} \tag{1.10}$$

Letting T tend to T_+ gives the estimate for $|N_+|$.

To prove the estimate (0.10), we simply replace $M(t_1)$ by E_1.

If we relax the conditions (0.6) and (0.7) to include the case $\epsilon_0 = 0$, a volume estimate is still possible.

Theorem 1.1. *If the assumptions of Theorem 0.1 are valid with $\epsilon_0 = 0$, and if in addition the length of any future directed curve starting from $M(t_1)$ is bounded by a constant γ_1 and the length of any past directed curve starting from $M(t_2)$ is bounded by a constant γ_2, then,*

$$|N_+| \le \gamma_1 |M(t_1)| \tag{1.11}$$

and

$$|N_-| \le \gamma_2 |M(t_2)|. \tag{1.12}$$

Proof. As before, we only consider the estimate for N_+.

From (1.6) we infer that the volume element of the slices $M(t)$ is decreasing in t, and hence,

$$\sqrt{g(t)} \le \sqrt{g(t_1)} \qquad \forall\, t_1 \le t. \tag{1.13}$$

Furthermore, for fixed $x \in M(t_1)$ and $t > t_1$

$$\int_{t_1}^{t} e^{\psi} \le \gamma_1 \tag{1.14}$$

because the left-hand side is the length of the future directed curve

$$\gamma(\tau) = (\tau, x) \qquad t_1 \le \tau \le t. \tag{1.15}$$

Let us now look at the cylinder $Q(t_1,T)$ as in (1.8) and (1.9). We have

$$\begin{aligned}|Q(t_1,T)| &= \int_{t_1}^T \int_{M(t_1)} e^{\psi}\sqrt{g(t,x)} \le \int_{t_1}^T \int_{M(t_1)} e^{\psi}\sqrt{g(t_1,x)} \\ &\le \gamma_1 \int_{M(t_1)} \sqrt{g(t_1,x)} = \gamma_1 |M(t_1)|\end{aligned} \tag{1.16}$$

by applying Fubini's theorem and the estimates (1.13) and (1.14). □

2. Cosmological spacetimes

A cosmological spacetime is a globally hyperbolic Lorentzian manifold N with compact Cauchy hypersurface $\mathcal{S}_0$, that satisfies the timelike convergence condition, i.e.

$$\bar R_{\alpha\beta}\nu^\alpha\nu^\beta \ge 0 \qquad \forall\, \langle \nu,\nu\rangle = -1. \tag{2.1}$$

If there exist crushing singularities, see [1] or [2] for a definition, then, we proved in [2] that N can be foliated by spacelike hypersurfaces $M(\tau)$ of constant mean curvature τ, $-\infty < \tau < \infty$,

$$N = \bigcup_{0\ne\tau\in\mathbb{R}} M(\tau) \cup \mathcal{C}_0, \tag{2.2}$$

where $\mathcal{C}_0$ consists either of a single maximal slice or of a whole continuum of maximal slices in which case the metric is stationary in $\mathcal{C}_0$. But in any case $\mathcal{C}_0$ is a compact subset of N.

In the complement of $\mathcal{C}_0$ the mean curvature function τ is a regular function with non-vanishing gradient that can be used as a new time function, cf. [4] for a simple proof.

Thus, the Lorentz metric can be expressed in Gaussian coordinates (x^α) with $x^0 = \tau$ as in (0.3). We choose arbitrary $\tau_2 < 0 < \tau_1$ and define

$$\begin{aligned} N_0 &= \{\, (\tau,x)\colon \tau_2 \le \tau \le \tau_1 \,\}, \\ N_- &= \{\, (\tau,x)\colon -\infty < \tau \le \tau_2 \,\}, \\ N_+ &= \{\, (\tau,x)\colon \tau_1 \le \tau < \infty \,\}. \end{aligned} \tag{2.3}$$

Then, N_0 is compact, and the volumes of N_-, N_+ can be estimated by

$$|N_+| \le \frac{1}{\tau_1}|M(\tau_1)|, \tag{2.4}$$

and

$$|N_-| \le \frac{1}{|\tau_2|}|M(\tau_2)|. \tag{2.5}$$

Hence, we have proved

Theorem 2.1. *A cosmological spacetime N with crushing singularities has finite volume.*

Remark 2.2. Let N be a spacetime with compact Cauchy hypersurface and suppose that a subset $N_- \subset N$ is foliated by constant mean curvature slices $M(\tau)$ such that

$$N_- = \bigcup_{0<\tau\le\tau_2} M(\tau) \tag{2.6}$$

and suppose furthermore, that $x^0 = \tau$ is a time function—which will be the case if the timelike convergence condition is satisfied—so that the metric can be represented in Gaussian coordinates (x^α) with $x^0 = \tau$.

Consider the cylinder $Q(\tau,\tau_2) = \{\tau \le x^0 \le \tau_2\}$ for some fixed τ. Then,

$$|Q(\tau,\tau_2)| = \int_\tau^{\tau_2}\int_M e^\psi = \int_\tau^{\tau_2} H^{-1}\int_M He^\psi, \tag{2.7}$$

and we obtain in view of (1.7)

$$\tau_2^{-1}\{|M(\tau)| - |M(\tau_2)|\} \le |Q(\tau,\tau_2)|, \tag{2.8}$$

and conclude further

$$\lim_{\tau\to 0}|M(\tau)| \le \tau_2|N_-| + |M(\tau_2)|, \tag{2.9}$$

i.e.

$$\lim_{\tau\to 0}|M(\tau)| = \infty \implies |N_-| = \infty. \tag{2.10}$$

3. The Riemannian case

Suppose that N is a Riemannian manifold that is decomposed as in (0.1) with metric

$$d\bar{s}^2 = e^{2\psi}\{{dx^0}^2 + \sigma_{ij}(x^0, x)dx^i dx^j\}. \tag{3.1}$$

The Gauß formula and the Weingarten equation for a hypersurface now have the form

$$x^\alpha_{ij} = -h_{ij}\nu^\alpha, \tag{3.2}$$

and

$$\nu^\alpha_i = h^k_i x^\alpha_k. \tag{3.3}$$

As default normal vector—if such a choice is possible—we choose the outward normal, which, in case of the coordinate slices $M(t) = \{x^0 = t\}$ is given by

$$(\nu^\alpha) = e^{-\psi}(1, 0, \dots, 0). \tag{3.4}$$

Thus, the coordinate slices are solutions of the evolution problem

$$\dot{x} = e^\psi \nu, \tag{3.5}$$

and, therefore,

$$\dot{g}_{ij} = 2e^\psi h_{ij}, \tag{3.6}$$

i.e. we have the opposite sign compared to the Lorentzian case leading to

$$\frac{d}{dt}|M(t)| = \int_M e^\psi H. \tag{3.7}$$

The arguments in Section 1 now yield

Theorem 3.1. (i) *Suppose there exists a positive constant ϵ_0 such that the mean curvature $H(t)$ of the slices $M(t)$ is estimated by*

$$H(t) \ge \epsilon_0 \qquad \forall\, t_1 \le t < T_+, \tag{3.8}$$

and

$$H(t) \le -\epsilon_0 \qquad \forall\, T_- < t \le t_2, \tag{3.9}$$

then

$$|N_+| \le \frac{1}{\epsilon_0} \lim_{t \to T_+} |M(t)|, \tag{3.10}$$

and

$$|N_-| \le \frac{1}{\epsilon_0} \lim_{t \to T_-} |M(t|. \tag{3.11}$$

(ii) *On the other hand, if the mean curvature H is negative in N_+ and positive in N_-, then, we obtain the same estimates as Theorem 0.1, namely,*

$$|N_+| \le \frac{1}{\epsilon_0} |M(t_1)|, \tag{3.12}$$

and

$$|N_-| \le \frac{1}{\epsilon_0} |M(t_2)|. \tag{3.13}$$

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