Conjugacy criteria for half-linear ODE
in theory of PDE
with generalized $p$-Laplacian
and mixed powers

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\[
\text{div} \left( A(x) \|\nabla y\|^{p-2} \nabla y \right) + \left\langle \tilde{b}(x), \|\nabla y\|^{p-2} \nabla y \right\rangle \\
+ c(x) |y|^{p-2}y + \sum_{i=1}^{m} c_i(x) |y|^{p_i-2}y = e(x),
\]

(E)

- \( x = (x_1, \ldots, x_n)_{i=1}^n \in \mathbb{R}^n, \ p > 1, \ p_i > 1, \)
- \( A(x) \) is elliptic \( n \times n \) matrix with differentiable components, \( c(x) \) and \( c_i(x) \) are Hölder continuous functions, \( \tilde{b}(x) = (b_1(x), \ldots, b_n(x)) \) is continuous \( n \)-vector function,
- \( \nabla = \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \right)_{i=1}^n \) and \( \text{div} = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n} \) is the usual nabla and divergence operators,
- \( q \) is a conjugate number to the number \( p \), i.e., \( q = \frac{p}{p-1} \),
- \( \langle \cdot, \cdot \rangle \) is the usual scalar product in \( \mathbb{R}^n \), \( \| \cdot \| \) is the usual norm in \( \mathbb{R}^n \), \( \| A \| = \sup \{ \| Ax \| : x \in \mathbb{R}^n \text{ with } \| x \| = 1 \} = \lambda_{\text{max}} \) is the spectral norm
- **solution** of (E) in \( \Omega \subseteq \mathbb{R}^n \) is a differentiable function \( u(x) \) such that
  - \( A(x) \|\nabla u(x)\|^{p-2} \nabla u(x) \) is also differentiable and \( u \) satisfies (E) in \( \Omega \)
- \( S(a) = \{ x \in \mathbb{R}^n : \| x \| = a \}, \)
  - \( \Omega(a) = \{ x \in \mathbb{R}^n : a \leq \| x \| \} \),
  - \( \Omega(a, b) = \{ x \in \mathbb{R}^n : a \leq \| x \| \leq b \} \)
Concept of oscillation for ODE

\[ u'' + c(x)u = 0 \] (1)

- Equation (1) is oscillatory if each solution has infinitely many zeros in \([x_0, \infty)\).
- Equation (1) is oscillatory if each solution has a zero \([a, \infty)\) for each \(a\).
- Equation (1) is oscillatory if each solution has conjugate points on the interval \([a, \infty)\) for each \(a\).
- All definition are equivalent (no accumulation of zeros and Sturm separation theorem).
- Equation is oscillatory if \(c(x)\) is large enough. Many oscillation criteria are expressed in terms of the integral \(\int_{\infty}^{\infty} c(x) \, dx\) (Hille and Nehari type)
- There are oscillation criteria which can detect oscillation even if \(\int_{\infty}^{\infty} c(x) \, dx\) is extremly small. These criteria are in fact series of conjugacy criteria.
Equation with mixed powers

\[
(p(t)u')' + c(t)u + \sum_{i=1}^{m} c_i(t)|u|^{\alpha_i} \text{sgn } u = e(t)
\]  

(2)

where \(\alpha_1 > \cdots > \alpha_m > 1 > \alpha_{m+1} > \cdots > \alpha_n > 0\).

**Theorem A** (Sun, Wong (2007)). If for any \(T \geq 0\) there exists \(a_1, b_1, a_2, b_2\) such that \(T \leq a_1 < b_1 \leq a_2 < b_2\) and

\[
\begin{aligned}
c_i(t) & \geq 0 & t \in [a_1, b_1] \cup [a_2, b_2], & i = 1, 2, \ldots, n \\
e(x) & \leq 0 & t \in [a_1, b_1] \\
e(x) & \geq 0 & t \in [a_2, b_2]
\end{aligned}
\]

and there exists a continuously differentiable function \(u(t)\) satisfying \(u(a_i) = u(b_i) = 0, u(t) \neq 0\) on \((a_i, b_i)\) and

\[
\int_{a_i}^{b_i} \left\{ p(t)u'^2(t) - Q(t)u^2(t) \right\} \, dt \leq 0
\]

(3)

for \(i = 1, 2\), where

\[
Q(t) = k_0 |e(t)|^{\eta_0} \prod_{i=1}^{m} \left( c_i^{\eta_i}(t) \right) + c(t),
\]

\(k_0 = \prod_{i=0}^{m} \eta_i^{-\eta_i} \) and \(\eta_i, i = 0, \ldots, n\) are positive constants satisfying \(\sum_{i=1}^{m} \alpha_i \eta_i = 1\) and \(\sum_{i=0}^{m} \eta_i = 1\),

then all solutions of (2) are oscillatory.
Concept of oscillation for linear PDE

\[ \Delta u + c(x)u = 0 \]  \hspace{1cm} (4)

- Equation (4) is *oscillatory* if every solution has a zero on \( \{ x \in \mathbb{R}^n : \|x\| \geq a \} \) for each \( a \).
- Equation (4) is *nodally oscillatory* if every solution has a nodal domain on \( \{ x \in \mathbb{R}^n : \|x\| \geq a \} \) for each \( a \).
- Both definitions are equivalent (Moss+Piepenbrink).

Concept of oscillation for half-linear PDE

\[ \text{div} \left( \| \nabla u \|^{p-2} \nabla u \right) + c(x)|u|^{p-2}u = 0 \] \hspace{1cm} (5)

- Essentially the same approach to oscillation as in linear case
- The equivalence between two oscillations is an open problem.
\[
\text{div} \left( A(x) \| \nabla y \|^{-2} \nabla y \right) + \left\langle \vec{b}(x), \| \nabla y \|^{-2} \nabla y \right\rangle \\
+ c(x) |y|^{-2} y + \sum_{i=1}^{m} c_i(x) |y|^{p_i-2} y = e(x),
\]

(E)

**Detection of oscillation from ODE**

**Theorem B** (O. Došlý (2001)). *Equation*

\[
\text{div}(\| \nabla u \|^{-2} \nabla u) + c(x) |u|^{-2} u = 0
\]

is oscillatory, if the ordinary differential equation

\[
\left( r^{n-1} |u'|^{-2} u' \right)' + r^{n-1} \left( \frac{1}{\omega_n r^{n-1}} \int_{S(r)} c(x) \, dx \right) |u|^{-2} u = 0
\]

(7)

is oscillatory. The number \( \omega_n \) is the surface area of the unit sphere in \( \mathbb{R}^n \).

J. Jaroš, T. Kusano and N. Yoshida proved independently similar result (for \( A(x) = a(\| x \|) I, a(\cdot) \) differentiable).

**Our aim**

- Extend method used in Theorem A to (E). Derive a general result, like Theorem B.
- Derive a result which does depend on more general expression, than the mean value of \( c(x) \) over spheres centered in the origin.
- Remove restrictions used by previous authors (for example Xu (2009) excluded the possibility \( p_i > p \) for every \( i \)).
\[
\text{div} \left( A(x) \| \nabla y \|^{p-2} \nabla y \right) + \left\langle \bar{b}(x), \| \nabla y \|^{p-2} \nabla y \right\rangle \\
+ c(x) |y|^{p-2}y + \sum_{i=1}^{m} c_i(x) |y|^{p_i-2}y = e(x),
\]

\hspace{10cm} \text{(E)}

**Modus operandi**

- Get rid of terms \( \sum_{i=1}^{m} c_i(x) |y|^{p_i-2}y \) and \( e(x) \) (join with \( c(x) |y|^{p-2}y \)) and convert the problem into
\[
\text{div} \left( A(x) \| \nabla y \|^{p-2} \nabla y \right) + \left\langle \bar{b}(x), \| \nabla y \|^{p-2} \nabla y \right\rangle + C(x) |y|^{p-2}y = 0.
\]

- Derive Riccati type inequality in \( n \) variables.

- Derive Riccati type inequality in 1 variable.

- Use this inequality as a tool which transforms results from ODE to PDE.
Using generalized AG inequality $\sum \alpha_i \geq \prod \left( \frac{\alpha_i}{\eta_i} \right)^{\eta_i}$, if $\alpha_i \geq 0, \eta_i > 0$ and $\sum \eta_i = 1$ we eliminate the right-hand side and terms with mixed powers.

**Lemma 1.** Let either $y > 0$ and $e(x) \leq 0$ or $y < 0$ and $e(x) \geq 0$. Let $\eta_i > 0$ be numbers satisfying $\sum_{i=0}^{m} \eta_i = 1$ and $\eta_0 + \sum_{i=1}^{m} p_i \eta_i = p$ and let $c_i(x) \geq 0$ for every $i$. Then

$$\frac{1}{|y|^{p-2}y} \left( -e(x) + \sum_{i=1}^{m} c_i(x) |y|^{p_i - 2}y \right) \geq C_1(x),$$

where

$$C_1(x) := \left| \frac{e(x)}{\eta_0} \right|^{\eta_0} \prod_{i=1}^{m} \left( \frac{c_i(x)}{\eta_i} \right)^{\eta_i}.$$  \hfill (8)

**Remark:** The numbers $\eta_i$ from Lemma 1 exist, if $p_i > p$ for some $i$.

**Lemma 2.** Suppose $c_i(x) \geq 0$. Let $\eta_i > 0$ be numbers satisfying $\sum_{i=1}^{m} \eta_i = 1$ and $\sum_{i=1}^{m} p_i \eta_i = p$. Then

$$\frac{1}{|y|^{p-2}y} \sum_{i=1}^{m} c_i(x) |y|^{p_i - 2}y \geq C_2(x),$$

where

$$C_2(x) := \prod_{i=1}^{m} \left( \frac{c_i(x)}{\eta_i} \right)^{\eta_i}.$$  \hfill (9)

**Remark:** The numbers $\eta_i$ from Lemma 2 exist iff $p_i > p$ for some $i$ and $p_j < p$ for some $j$. 
Lemma 3. Let $y$ be a solution of $(E)$ which does not have zero on $\Omega$. Suppose that there exists a function $C(x)$ such that

$$C(x) \leq c(x) + \sum_{i=1}^{m} c_i(x) |y|^{p_i - p} - \frac{e(x)}{|y|^{p-2} y}$$

Denote $\vec{w}(x) = A(x) \frac{\|\nabla y\|^{p-2} \nabla y}{|y|^{p-2} y}$. The function $\vec{w}(x)$ is well defined on $\Omega$ and satisfies the inequality

$$\text{div} \vec{w} + (p - 1) \Lambda(x) \|\vec{w}\|^q + \left\langle \vec{w}, A^{-1}(x) \vec{b}(x) \right\rangle + C(x) \leq 0 \quad (10)$$

where

$$\Lambda(x) = \begin{cases} \lambda_{\max}^{1-q}(x), & 1 < p \leq 2, \\ \lambda_{\min} \lambda_{\max}^{-q}(x), & p > 2. \end{cases} \quad (11)$$

Lemma 4. Let $(10)$ hold. Let $l > 1$, $l^* = \frac{l}{l - 1}$ be two mutually conjugate numbers and $\alpha \in C^1(\Omega, \mathbb{R}^+)$ be a smooth function positive on $\Omega$. Then

$$\text{div}(\alpha(x) \vec{w}) + (p - 1) \frac{\Lambda(x) \alpha^{1-q}(x)}{l^*} \|\alpha(x) \vec{w}\|^q - l^{p-1} \frac{\alpha(x)}{p^p \Lambda^{p-1}(x)} \left\|A^{-1}(x) \vec{b}(x) - \frac{\nabla \alpha(x)}{\alpha(x)} \right\|^p + \alpha(x) C(x) \leq 0$$

holds on $\Omega$. If $\left\|A^{-1} \vec{b} - \frac{\nabla \alpha}{\alpha} \right\| \equiv 0$ on $\Omega$, then this inequality holds with $l^* = 1$. 

CDDEA 2010, Rajecké Teplice (9/12)
**Theorem 1.** Let the \( n \)-vector function \( \bar{w} \) satisfy inequality
\[
\text{div} \bar{w} + C_0(x) + (p - 1) \Lambda_0(x) \| \bar{w} \|^q \leq 0
\]
on \( \Omega(a,b) \). Denote \( \tilde{C}(r) = \int_{S(r)} C_0(x) \, d\sigma \) and \( \tilde{R}(r) = \int_{S(r)} \Lambda_0^{1-p} \, d\sigma \). Then the half-linear ordinary differential equation
\[
\left( \tilde{R}(r) |u'|^{p-2} u' \right)' + \tilde{C}(r) |u|^{p-2} u = 0, \quad \' = \frac{d}{dr}
\]
is disconjugate on \([a,b]\) and it possesses solution which has no zero on \([a,b]\).

**Theorem 2.** Let \( l > 1 \). Let \( l^* = 1 \) if \( \| \bar{b} \| \equiv 0 \) and \( l^* = \frac{l}{l-1} \) otherwise. Further, let \( c_i(x) \geq 0 \) for every \( i \). Denote
\[
\tilde{R}(r) = (l^*)^{p-1} \int_{S(r)} \Lambda^{1-p}(x) \, d\sigma
\]
and
\[
\tilde{C}(r) = \int_{S(r)} c(x) + C_1(x) - \frac{l^{p-1}}{p \Lambda^{p-1}(x)} \left\| A^{-1}(x) \bar{b}(x) \right\|^p \, d\sigma,
\]
where \( \Lambda(x) \) is defined by (11) and \( C_1(x) \) is defined by (8).
Suppose that the equation
\[
\left( \tilde{R}(r) |u'|^{p-2} u' \right)' + \tilde{C}(r) |u|^{p-2} u = 0
\]
has conjugate points on \([a,b]\).
If \( e(x) \leq 0 \) on \( \Omega(a,b) \), then equation (E) has no positive solution on \( \Omega(a,b) \).
If \( e(x) \geq 0 \) on \( \Omega(a,b) \), then equation (E) has no negative solution on \( \Omega(a,b) \).
Theorem 3 (non-radial variant of Theorem 2). Let $l > 1$ and let $\Omega \subset \Omega(a,b)$ be an open domain with piecewise smooth boundary such that $\text{meas}(\Omega \cap S(r)) \neq 0$ for every $r \in [a,b]$. Let $c_i(x) \geq 0$ on $\Omega$ for every $i$ and let $\alpha(x)$ be a function which is positive and continuously differentiable on $\Omega$ and vanishes on the boundary and outside $\Omega$. Let $l^* = 1$ if \[ \left\| A^{-1}b - \frac{\nabla \alpha}{\alpha} \right\| \equiv 0 \] on $\Omega$ and $l^* = \frac{l}{l-1}$ otherwise. In the former case suppose also that the integral

\[ \int_{S(r)} \frac{\alpha(x)}{\Lambda^{p-1}(x)} \left\| A^{-1}(x)b(x) - \frac{\nabla \alpha(x)}{\alpha(x)} \right\|^p \, d\sigma \]

which may have singularity on $\partial \Omega$ if $\Omega \neq \Omega(a,b)$ is convergent for every $r \in [a,b]$. Denote

\[ \tilde{R}(r) = (l^*)^{p-1} \int_{S(r)} \alpha(x)\Lambda^{1-p}(x) \, d\sigma \]

and

\[ \tilde{C}(r) = \int_{S(r)} \alpha(x) \left( c(x) + C_1(x) - \frac{l^{p-1}}{p^p\Lambda^{p-1}(x)} \left\| A^{-1}(x)b(x) - \frac{\nabla \alpha(x)}{\alpha(x)} \right\|^p \right) \, d\sigma, \]

where $\Lambda(x)$ is defined by (11) and $C_1(x)$ is defined by (8) and suppose that equation

\[ \left( \tilde{R}(r)|u'|^{p-2}u' \right)' + \tilde{C}(r)|u|^{p-2}u = 0 \]

has conjugate points on $[a,b]$.

If $e(x) \leq 0$ on $\Omega(a,b)$, then equation (E) has no positive solution on $\Omega(a,b)$.

If $e(x) \geq 0$ on $\Omega(a,b)$, then equation (E) has no negative solution on $\Omega(a,b)$.
Theorem 4. Let $l$, $\Omega$, $\alpha(x)$, $\Lambda(x)$ and $\tilde{R}(r)$ be defined as in Theorem 3 and let $c_i(x) \geq 0$ and $e(x) \equiv 0$ on $\Omega(a,b)$. Denote

$$\tilde{C}(r) = \int_{S(r)} \alpha(x) \left( c(x) + C_2(x) - \frac{l^{p-1}}{p^p \Lambda^{p-1}(x)} \left\| A^{-1}(x) \tilde{b}(x) - \frac{\nabla \alpha(x)}{\alpha(x)} \right\|^p \right) \, \, \, d\sigma,$$

where $C_2(x)$ is defined by (9). If the equation

$$\left( \tilde{R}(r)|u'|^{p-2}u' \right)' + \tilde{C}(r)|u|^{p-2}u = 0$$

has conjugate points on $[a,b]$, then every solution of equation (E) has zero on $\Omega(a,b)$.

Similar theorems can be derived also for estimates of terms with mixed powers based on different methods than AG inequality (see R. M., Nonlinear Analysis TMA 73 (2010)).