$$\begin{aligned}\langle P_{i,j}P_{k,l}\rangle &= \exp(4\langle\ell\rangle)\iint d\mathbf{r}\, d\mathbf{r}'\, W(r/R)\, W(r'/R) \\ &\quad\times \left\langle \exp[2\ell(\mathbf{r}+\boldsymbol{\rho}_i;\mathbf{r}_j) - 2\langle\ell\rangle + 2\ell(\mathbf{r}'+\boldsymbol{\rho}_k;\mathbf{r}_l) - 2\langle\ell\rangle]\right\rangle \qquad \text{(A10)} \\ &= \exp(4\langle\ell\rangle)\iint d\mathbf{r}\, d\mathbf{r}'\, W(r/R)\, W(r'/R) \\ &\quad\times \exp\{2[2\sigma_\ell^2 + 2\langle[\ell(\mathbf{r}+\boldsymbol{\rho}_i;\mathbf{r}_j) - \langle\ell\rangle][\ell(\mathbf{r}'+\boldsymbol{\rho}_k;\mathbf{r}_l) - \langle\ell\rangle]\rangle]\} \qquad \text{(A11)} \\ \Longrightarrow \langle P_{i,j}P_{k,l}\rangle &= \iint d\mathbf{r}\, d\mathbf{r}'\, W(r/R)\, W(r'/R) \\ &\quad\times \exp\{4\langle[\ell(\mathbf{r}+\boldsymbol{\rho}_i;\mathbf{r}_j) - \langle\ell\rangle][\ell(\mathbf{r}'+\boldsymbol{\rho}_k;\mathbf{r}_l) - \langle\ell\rangle]\rangle\} \qquad \text{(A12)}\end{aligned}$$

We wish to develop an expression where $\langle(P_{i,j} - P_{k,l})^2\rangle$ is linearly related to $C_n^2(z)$, so that matrix algebra can be used to estimate the $C_n^2(z)$ profile. This can be achieved by expanding the exponent in Eq. (A12) with $\exp(x) \approx 1 + x$. This two term expansion is also consistent with Rytov theory. Expanding the integrand in Eq. (A12) yields

$$\begin{aligned}\langle P_{i,j}P_{k,l}\rangle &\approx \pi^2R^4 + 4\iint d\mathbf{r}\, d\mathbf{r}'\, W(r/R)\, W(r'/R) \\ &\quad\times \langle[\ell(\mathbf{r}+\boldsymbol{\rho}_i;\mathbf{r}_j) - \langle\ell\rangle][\ell(\mathbf{r}'+\boldsymbol{\rho}_k;\mathbf{r}_l) - \langle\ell\rangle]\rangle.\end{aligned} \qquad \text{(A13)}$$

It is straightforward to reduce Eq. (A13) using the following well-known expression for ℓ:

$$\begin{aligned}\ell(\mathbf{u};\mathbf{v}) - \langle\ell\rangle &= k\int_0^L dz \iint d\boldsymbol{\kappa}_t\, d\kappa_z\, \tilde{n}(\boldsymbol{\kappa}_t,\kappa_z)\exp(-i2\pi\kappa_z z) \\ &\quad\times \exp(-i2\pi q\boldsymbol{\kappa}_t\cdot\mathbf{u})\exp\left(-i2\pi\frac{z}{L}\boldsymbol{\kappa}_t\cdot\mathbf{v}\right)\sin(\pi\lambda zq\kappa_t^2).\end{aligned} \qquad \text{(A14)}$$

The terms λ, k, and $\tilde{n}$ in Eq. (A14) are the wavelength of the radiation, the wavenumber ($k = 2\pi/\lambda$), and the Fourier transform of the index of refraction profile. It is related to the index of refraction, $n(\mathbf{s}, z)$ by

$$n(\mathbf{s}, z) = \iint d\boldsymbol{\kappa}_t\, d\kappa_z\, \tilde{n}(\boldsymbol{\kappa}_t,\kappa_z)\exp(-i2\pi\kappa_z z)\exp(-i2\pi\boldsymbol{\kappa}_t\cdot\mathbf{s}). \qquad \text{(A15)}$$

In the above expression, $n(\mathbf{s}, z)$ is the index of refraction at range z from the pupil plane and at transverse coordinate $\mathbf{s}$. The variables $\boldsymbol{\kappa}_t$ and κ_z are the transverse and longitudinal spatial frequencies of the index of refraction. The integration in Eq. (A14) ranges from the pupils at $z = 0$ to the beacons at $z = L$. Finally, q is shorthand notation for

$$q = 1 - z/L. \qquad \text{(A16)}$$

As a side note, the corresponding expression for the Rytov phase is also given by Eq. (A14), but with the sin term replaced with cos. In the geometric optics limit, the argument of the cos term, $\pi\lambda zq\kappa_t^2$, is small so $\cos(\pi\lambda zq\kappa_t^2)$ can be approximated by unity and we find the geometric optics expression for phase is

$$\begin{aligned}\phi(\mathbf{u};\mathbf{v}) &= k\int_0^L dz \iint d\boldsymbol{\kappa}_t\, d\kappa_z\, \tilde{n}(\boldsymbol{\kappa}_t,\kappa_z)\exp(-i2\pi\kappa_z z)\exp\left[-i2\pi\boldsymbol{\kappa}_t\cdot\left(q\mathbf{u} + \frac{z}{L}\mathbf{v}\right)\right] \qquad \text{(A17)} \\ &= k\int_0^L dz\, n\left[(1 - z/L)\mathbf{u} + \frac{z}{L}\mathbf{v}, z\right]. \qquad \text{(A18)}\end{aligned}$$

Equation (A18) is a fundamental relationship between the index of refraction profile and ϕ. It was used as a starting point in the derivation of the differential tilt covariance expression in Ref. 1.